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THE ASYMPTOTIC SOLUTION OF THE THREE-DIMENSIONAL PROBLEM OF THE THEORY OF ELASTICITY FOR PLATES OF INCOMPRESSIBLE MATERIALS†

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The asymptotic form is established and recurrence formulae are derived for determining the stress–strain state of plates of incompressible materials, when the kinematic conditions are specified on the face surfaces of the plate. By combining the solution obtained with the solution of similar problems for plates of compressible materials, a solution of the boundary-value problem for a three-layer plate with an incompressible middle layer is constructed when there is complete contact between the layers. Examples are given which model, in particular, the operation of rubber–metal seismic isolators.

The asymptotic method [1–5] has been found to be effective for solving boundary-value problems of the theory of elasticity for multilayered plates and shells, when the kinematic and mixed boundary conditions are specified on their face surfaces (non-classical boundary-value problems of the theory of plates). However, the recurrence formulae established in [1–5] for plates of compressible materials contain a singularity in the case of incompressible materials, which makes them unsuitable for determining the stress–strain state of plates of incompressible materials.

1. SOLUTION OF THE INNER PROBLEM

We consider an isotropic thermoelastic thin plate of incompressible material, which occupies the region

$$\Omega = \{x, y, z:\ 0 \leqslant x \leqslant a,\ 0 \leqslant y \leqslant b,\ -h \leqslant z \leqslant h,\ h \ll l,\ l = \min\{a, b\}\}$$

It is required to determine the stress–strain state of the plate if the components of the displacement vector

$$u_j(z = \pm h) = u_j^{\pm}(x, y), \quad j = x, y, z \tag{1.1}$$

are specified on the face surfaces, while one of the groups of classical boundary conditions of the theory of elasticity is specified on the side surface. It is also assumed that there is a temperature field present which is taken into account using the Duhamel–Neumann model.

To solve this boundary-value problem we change to dimensionless coordinates and dimensionless displacements

$$\xi = \frac{x}{l}, \quad \eta = \frac{y}{l}, \quad \zeta = \frac{z}{h} = \varepsilon^{-1}\frac{z}{l}\left(\varepsilon = \frac{h}{l}\right), \quad u = \frac{u_x}{l}, \quad v = \frac{u_y}{l}, \quad w = \frac{u_z}{l} \tag{1.2}$$

in the equations of equilibrium of the three-dimensional problem of the theory of elasticity and the elasticity relations, taking into account the temperature deformations and incompressibility of the material.

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As a result we obtain the following system of equations, singularly perturbed by a small parameter ε

$$\frac{\partial\sigma_{xx}}{\partial\xi}+\frac{\partial\sigma_{xy}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{xz}}{\partial\zeta}=0\quad (x,y;\ \xi,\eta),\qquad \frac{\partial\sigma_{xz}}{\partial\xi}+\frac{\partial\sigma_{yz}}{\partial\eta}+\varepsilon^{-1}\frac{\partial\sigma_{zz}}{\partial\zeta}=0$$

$$\sigma_{xx}=\sigma_{zz}+2G\left(2\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}\right)-6G\alpha\theta\quad (x,y;\ \xi,\eta;\ u,v),\qquad \sigma_{xy}=G\left(\frac{\partial u}{\partial\eta}+\frac{\partial v}{\partial\xi}\right)$$

$$\varepsilon^{-1}\frac{\partial u}{\partial\zeta}=\frac{1}{G}\sigma_{xz}-\frac{\partial w}{\partial\xi}\quad (x,y;\ \xi,\eta;\ u,v) \tag{1.3}$$

$$\varepsilon^{-1}\frac{\partial w}{\partial\zeta}=-\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}\right)+3\alpha\theta,\qquad \varepsilon^{-1}\frac{\partial w}{\partial\zeta}=\frac{1}{3G}\left[\sigma_{zz}-\frac{1}{2}(\sigma_{xx}+\sigma_{yy})\right]+\alpha\theta$$

(the penultimate equation takes into account the incompressibility condition).

It is easy to verify that when the incompressibility condition and the remaining elasticity relations are satisfied, the last of relations (1.3) is satisfied automatically, and hence it will not be considered in what follows.

The solution of system (1.3) will be sought in the form of the asymptotic expansion

$$Q=\varepsilon^{\chi_Q+s}Q^{(s)}(\xi,\eta,\zeta),\quad s=\overline{0,S} \tag{1.4}$$

where Q is any of the required quantities, χ_Q represents the asymptotic order of the corresponding quantity, and $s=\overline{0,S}$ denotes that summation is carried out over the dummy index in the limits indicated. The quantities χ_Q must be chosen in such a way that, after substituting series (1.4) into system (1.3), a consistent system in $Q^{(s)}$ is obtained. This is achieved only when

$$\chi_{\sigma_{xx}}=\chi_{\sigma_{yy}}=\chi_{\sigma_{zz}}=-3,\quad \chi_{\sigma_{xz}}=\chi_{\sigma_{yz}}=\chi_{\sigma_{xy}}=-2,\quad \chi_{u_x}=\chi_{u_y}=-1,\quad \chi_{u_z}=0 \tag{1.5}$$

The asymptotic form (1.5) differs in principle from the asymptotic forms [1–5] obtained for strips and plates of compressible materials in the case of both the classical and non-classical boundary-value problem.

It is assumed that the contribution of the temperature field is commensurable with the contribution of the surface forces, for which it is necessary that

$$\theta=\varepsilon^{-1+s}\theta^{(s)}(\xi,\eta,\zeta),\quad s=\overline{0,S} \tag{1.6}$$

Substituting expressions (1.4)–(1.6) into system (1.3), we obtain a consistent system of equations in the expansion coefficients of $Q^{(s)}$. Its solution has the form

$$\sigma_{zz}^{(s)}=\sigma_{zz0}^{(s)}(\xi,\eta)+\sigma_{zz*}^{(s)}(\xi,\eta,\zeta) \tag{1.7}$$

$$\sigma_{xx}^{(s)}=\sigma_{zz}^{(s)}+2G\left(2\frac{\partial u^{(s-2)}}{\partial\xi}+\frac{\partial v^{(s-2)}}{\partial\eta}\right)-6G\alpha\theta^{(s-2)}\quad (x,y;\ \xi,\eta;\ u,v)$$

$$\sigma_{xy}^{(s)}=G\left(\frac{\partial u^{(s-1)}}{\partial\eta}+\frac{\partial v^{(s-1)}}{\partial\xi}\right);\quad \sigma_{xz}^{(s)}=\sigma_{xz0}^{(s)}(\xi,\eta)-\zeta\frac{\partial\sigma_{zz0}^{(s)}}{\partial\xi}+\sigma_{xz*}^{(s)}(\xi,\eta,\zeta)\quad (x,y;\ \xi,\eta)$$

$$u^{(s)}=u_0^{(s)}(\xi,\eta)+\frac{1}{G}\zeta\sigma_{xz0}^{(s)}-\frac{1}{2G}\zeta^2\frac{\partial\sigma_{zz0}^{(s)}}{\partial\xi}+u_*^{(s)}(\xi,\eta,\zeta)\quad (x,y;\ \xi,\eta;\ u,v)$$

$$w^{(s)}=w_0^{(s)}(\xi,\eta)-\zeta\left(\frac{\partial u_0^{(s)}}{\partial\xi}+\frac{\partial v_0^{(s)}}{\partial\eta}\right)-\frac{\zeta^2}{2G}\left(\frac{\partial\sigma_{xz0}^{(s)}}{\partial\xi}+\frac{\partial\sigma_{yz0}^{(s)}}{\partial\eta}\right)+$$

$$+\frac{\zeta^3}{6G}\left(\frac{\partial^2\sigma_{zz0}^{(s)}}{\partial\xi^2}+\frac{\partial^2\sigma_{zz0}^{(s)}}{\partial\eta^2}\right)+w_*(\xi,\eta,\zeta)$$

$$\sigma_{zz^*}^{(s)} = -\int_0^\zeta \left(\frac{\partial \sigma_{xz}^{(s-2)}}{\partial \xi} + \frac{\partial \sigma_{yz}^{(s-2)}}{\partial \eta} \right) d\zeta$$

$$\sigma_{xz^*}^{(s)} = -\int_0^\zeta \left[\frac{\partial \sigma_{zz^*}^{(s)}}{\partial \xi} + G\left(\left(4\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right) u^{(s-2)} + 3\frac{\partial^2 v^{(s-2)}}{\partial \xi \partial \eta} - 6\alpha\theta^{(s-2)} \right) \right] d\zeta \quad (x, y;\ \xi, \eta;\ u, v)$$

$$u_*^{(s)} = \int_0^\zeta \left(\frac{1}{G}\sigma_{xz^*}^{(s)} - \frac{\partial w^{(s-2)}}{\partial \xi} \right) d\zeta \quad (x, y;\ \xi, \eta;\ u, v), \quad w_*^{(s)} = -\int_0^\zeta \left(\frac{\partial u_*^{(s)}}{\partial \xi} + \frac{\partial v_*^{(s)}}{\partial \eta} - 3\alpha\theta^{(s)} \right) d\zeta$$

where $\sigma_{xz0}^{(s)}$, $\sigma_{yz0}^{(s)}$, $\sigma_{zz0}^{(s)}$, $u_0^{(s)}$, $v_0^{(s)}$ and $w_0^{(s)}$ are as yet unknown functions of the integration, which will be determined when satisfying the boundary conditions.

By satisfying conditions (1.1) and reverting to dimensional coordinates, we obtain the following recurrence formulae for determining the components of the stress tensor and the displacement vector

$$R = \sum_{s=0}^{S} R^{(s)}(x, y, z)$$

$$\sigma_{zz}^{(s)} = \sigma^{(s)}(x, y) + \sigma_{zz^*}^{(s)}(x, y, z)$$

$$\sigma_{xx}^{(s)} = \sigma_{zz}^{(s)} + 2G\left(2\frac{\partial u_x^{(s-2)}}{\partial x} + \frac{\partial u_y^{(s-2)}}{\partial y} \right) - 6G\alpha\theta^{(s-2)} \quad (x, y)$$

$$\sigma_{xy}^{(s)} = G\left(\frac{\partial u_x^{(s-1)}}{\partial y} + \frac{\partial u_y^{(s-1)}}{\partial x} \right); \quad \sigma_{xz}^{(s)} = \frac{G}{2h}U_x^{(s)} + \sigma_{zx^*}^{(s)} - z\frac{\partial \sigma^{(s)}}{\partial x} \quad (x, y)$$

$$u_x^{(s)} = u_x^{-(s)} + u_{x^*}^{(s)} + \frac{z+h}{2h}U_x^{(s)} + \frac{1}{2G}(h^2 - z^2)\frac{\partial \sigma^{(s)}}{\partial x} - u_{x^*}^{(s)}(z=-h) \quad (x, y) \tag{1.8}$$

$$u_z^{(s)} = u_z^{-(s)} + u_{z^*}^{(s)} - \frac{(z+h)^2}{4h}\left(\frac{\partial U_x^{(s)}}{\partial x} + \frac{\partial U_y^{(s)}}{\partial y} \right) + \frac{(2h-z)(h+z)^2}{4h^3}W^{(s)} - u_{z^*}^{(s)}(z=-h) -$$

$$-(z+h)\left[\frac{\partial}{\partial x}(u_x^{-(s)} - u_{x^*}^{(s)}(z=-h)) + \frac{\partial}{\partial y}(u_y^{-(s)} - u_{y^*}^{(s)}(z=-h)) \right]$$

$$\sigma_{zz^*}^{(s)} = -\int_0^z \left(\frac{\partial \sigma_{xz}^{(s-2)}}{\partial x} + \frac{\partial \sigma_{yz}^{(s-2)}}{\partial y} \right) dz$$

$$\sigma_{xz^*}^{(s)} = -\int_0^z \left[\frac{\partial \sigma_{zz^*}^{(s)}}{\partial x} + G\left(4\frac{\partial^2 u_x^{(s-2)}}{\partial x^2} + \frac{\partial^2 u_x^{(s-2)}}{\partial y^2} + 3\frac{\partial^2 u_y^{(s-2)}}{\partial x \partial y} - 6\alpha\frac{\partial \theta^{(s-2)}}{\partial x} \right) \right] dz \quad (x, y)$$

$$u_{x^*}^{(s)} = \int_0^z \left(\frac{1}{G}\sigma_{xz^*}^{(s)} - \frac{\partial u_z^{(s-2)}}{\partial x} \right) dz \quad (x, y), \quad u_{z^*}^{(s)} = -\int_0^z \left(\frac{\partial u_{x^*}^{(s)}}{\partial x} + \frac{\partial u_{y^*}^{(s)}}{\partial y} - 3\alpha\theta^{(s)} \right) dz$$

$$U_x^{(s)} = u_x^{+(s)} - u_x^{-(s)} - u_{x^*}^{(s)}(z=h) + u_{x^*}^{(s)}(z=-h) \quad (x, y, z)$$

$$W^{(s)} = U_z^{(s)} + h\left(\frac{\partial U_x^{(s)}}{\partial x} + \frac{\partial U_y^{(s)}}{\partial y} \right) + 2h\left[\frac{\partial}{\partial x}(u_x^{-(s)} - u_{x^*}^{(s)}(z=-h)) + \frac{\partial}{\partial y}(u_y^{-(s)} - u_{y^*}^{(s)}(z=-h)) \right]$$

$$u_x^{\pm(0)} = u_x^{\pm}, \quad u_x^{\pm(s)} = 0, \quad s \neq 0 \quad (x, y, z)$$

where $\sigma^{(s)}$ is the solution of Poisson's equation

$$\frac{\partial^2\sigma^{(s)}}{\partial x^2}+\frac{\partial^2\sigma^{(s)}}{\partial y^2}=-\frac{3G}{2h^3}W^{(s)} \tag{1.9}$$

In particular, for the principal (zeroth) approximation we have

$$W^{(0)}=u_z^+-u_z^-+h\left[\frac{\partial}{\partial x}(u_x^++u_z^-)+\frac{\partial}{\partial y}(u_y^++u_y^-)\right]-3\alpha\int_{-h}^{h}\theta^{(0)}dz \tag{1.10}$$

Note that in the case of a plate of compressible material the solution of the inner problem with boundary conditions (1.1) is completely determined after satisfying these conditions, and the conditions on the side surface have no effect on this solution – they specify the boundary layer. This specific feature of the solution was established for the first time for a strip in [6] and for plates in [2, 5, 7]. In this connection note also paper [8]. For plates of incompressible material, as we will see, a different picture has been obtained. Since the solution of the inner problem is expressed in terms of the solution of Eq. (1.9), which will contain new arbitrary constants to be determined from the conditions on the side surface, then, unlike the case of a compressible plate, the last conditions will naturally affect the solution of the internal problem.

By determining the solution of Eq. (1.9), we can determine all the required quantities of the inner problem for an arbitrary approximation of s from (1.8).

In certain cases, for example, when the functions specified on the face surfaces $z = \pm h$, are polynomials, the iterative process terminates abruptly and we arrive at a mathematically exact solution of the equations for a layer.

In view of the singular perturbability of the initial boundary-value problem, it is impossible to use the solution constructed to satisfy the boundary conditions at each point of the side surface of the plate. To do this it is necessary to construct the boundary layer. The solution of the boundary layer is constructed and matched with the solution of the inner problem in the well-known way [5, 9, 10].

2. SOLUTION OF THE BOUNDARY-LAYER PROBLEM

In order to construct a solution for the boundary layer, localized in the region of the edge $x = 0$, we make a new replacement of variable $t = \xi/\varepsilon$ in Eqs (1.3), and the solution of the newly obtained system, without retaining the temperature terms (they are taken into account in the solution of the inner problem), will be sought in the form

$$\begin{gathered}\sigma_{ik}=\varepsilon^{-1+s}\sigma_{ik}^{(s)}(\eta,\zeta)\exp(-\lambda t),\quad i,k=1,2,3\\ U,V,W=\varepsilon^s(U^{(s)},V^{(s)},W^{(s)})\exp(-\lambda t),\quad s=\overline{0,N}\end{gathered} \tag{2.1}$$

where σ_{ik}, U, V and W are the components of the stress tensor and the dimensionless displacement vector of the boundary layer, respectively, and λ is an as yet unknown number; Re $\lambda > 0$ represents the rate of attenuation of the values of the boundary layer with distance from the edge. Substituting expressions (2.1) into the converted system (1.3), we obtain the following two subsystems for determining the coefficients of the expansion (2.1).

$$-\lambda\sigma_{12}^{(s)}+\frac{\partial\sigma_{23}^{(s)}}{\partial\zeta}+\frac{\partial\sigma_{22}^{(s-1)}}{\partial\eta}=0,\quad \sigma_{12}^{(s)}=-G\lambda V^{(s)}+G\frac{\partial U^{(s-1)}}{\partial\eta}, \tag{2.2}$$

$$\frac{\partial V^{(s)}}{\partial\zeta}+\frac{\partial W^{(s-1)}}{\partial\eta}=\frac{1}{G}\sigma_{23}^{(s)}$$

$$-\lambda\sigma_{11}^{(s)}(\eta,\zeta)+\frac{\partial\sigma_{13}^{(s)}}{\partial\zeta}+\frac{\partial\sigma_{12}^{(s-1)}}{\partial\eta}=0,\quad -\lambda\sigma_{13}^{(s)}+\frac{\partial\sigma_{33}^{(s)}}{\partial\zeta}+\frac{\partial\sigma_{23}^{(s-1)}}{\partial\eta}=0$$

$$\sigma_{11}^{(s)}=\sigma_{33}^{(s)}+2G\left(-2\lambda U^{(s)}+\frac{\partial V^{(s-1)}}{\partial\eta}\right),\quad \sigma_{22}^{(s)}=\sigma_{33}^{(s)}+2G\left(2\frac{\partial V^{(s-1)}}{\partial\eta}-\lambda U^{(s)}\right) \tag{2.3}$$

$$\frac{\partial U^{(s)}}{\partial\zeta}-\lambda W^{(s)}=\frac{1}{G}\sigma_{13}^{(s)},\quad \frac{\partial W^{(s)}}{\partial\zeta}-\lambda U^{(s)}+\frac{\partial V^{(s-1)}}{\partial\eta}=0$$

In system (2.2) the stresses can be expressed in terms of the displacements $V^{(s)}$

$$\sigma_{23}^{(s)}=G\frac{\partial V^{(s)}}{\partial\zeta}+G\frac{\partial W^{(s-1)}}{\partial\eta},\quad \sigma_{12}^{(s)}=-G\lambda V^{(s)}+G\frac{\partial U^{(s-1)}}{\partial\eta} \tag{2.4}$$

while $V^{(s)}$ is found from the equation

$$\frac{\partial^2 V^{(s)}}{\partial\zeta^2}+\lambda^2 V^{(s)}=R_v^{(s-1)};\quad R_v^{(s-1)}=\lambda\frac{\partial U^{(s-1)}}{\partial\eta}-\frac{\partial^2 W^{(s-1)}}{\partial\eta\partial\zeta}-\frac{1}{G}\frac{\partial\sigma_{22}^{(s-1)}}{\partial\eta} \tag{2.5}$$

The unknown quantities occurring in system (2.3) can be expressed in terms of $W^{(s)}$

$$\sigma_{11}^{(s)}=\frac{G}{\lambda^2}\left[\frac{\partial^3 W^{(s)}}{\partial\zeta^3}-\lambda^2\frac{\partial W^{(s)}}{\partial\zeta}\right]+R_{11}^{(s-1)},\quad \sigma_{22}^{(s)}=\frac{G}{\lambda^2}\left[\frac{\partial^3 W^{(s)}}{\partial\zeta^3}+\lambda^2\frac{\partial W^{(s)}}{\partial\zeta}\right]+R_{22}^{(s-1)}$$

$$\sigma_{33}^{(s)}=\frac{G}{\lambda^2}\left[\frac{\partial^3 W^{(s)}}{\partial\zeta^3}+3\lambda^2\frac{\partial W^{(s)}}{\partial\zeta}\right]+R_{33}^{(s-1)},\quad \sigma_{13}^{(s)}=\frac{G}{\lambda}\left[\frac{\partial^2 W^{(s)}}{\partial\zeta^2}-\lambda^2 W^{(s)}\right]+R_{13}^{(s-1)}$$

$$U^{(s)}=\frac{1}{\lambda}\left[\frac{\partial W^{(s)}}{\partial\zeta}+\frac{\partial V^{(s-1)}}{\partial\eta}\right] \tag{2.6}$$

$$R_{11}^{(s-1)}=\frac{1}{\lambda}\left[\frac{\partial\sigma_{12}^{(s-1)}}{\partial\eta}+\frac{G}{\lambda}\frac{\partial^3 V^{(s-1)}}{\partial\eta\partial\zeta^2}\right],\quad R_{22}^{(s-1)}=R_{33}^{(s-1)}+2G\frac{\partial V^{(s-1)}}{\partial\eta}$$

$$R_{33}^{(s-1)}=\frac{1}{\lambda}\frac{\partial\sigma_{12}^{(s-1)}}{\partial\eta}+\frac{G}{\lambda^2}\left[\frac{\partial^3 V^{(s-1)}}{\partial\eta\partial\zeta^2}+2\lambda^2\frac{\partial V^{(s-1)}}{\partial\eta}\right],\quad R_{13}^{(s-1)}=\frac{G}{\lambda}\frac{\partial^2 V^{(s-1)}}{\partial\eta\partial\zeta}$$

while to determine $W^{(s)}$ from system (2.3) we have the equation

$$\frac{\partial^4 W^{(s)}}{\partial\zeta^4}+2\lambda^2\frac{\partial^2 W^{(s)}}{\partial\zeta^2}+\lambda^4 W^{(s)}=R_w^{(s-1)}$$

$$R_w^{(s-1)}=-\frac{\partial^4 V^{(s-1)}}{\partial\eta\partial\zeta^3}-\lambda^2\frac{\partial^2 V^{(s-1)}}{\partial\eta\partial\zeta}-\frac{\lambda}{G}\left(\frac{\partial^2\sigma_{12}^{(s-1)}}{\partial\eta\partial\zeta}+\lambda\frac{\partial\sigma_{23}^{(s-1)}}{\partial\eta}\right) \tag{2.7}$$

Relations (2.4) and Eq. (2.5) define the antiplace boundary layer, while (2.6) and (2.7) define the plane boundary layer, i.e. the antiplane and plane boundary stress–strain states.

When $s = 0$, the right-hand sides of Eqs (2.5) and (2.7) vanish. Solving these homogeneous equations and satisfying the homogeneous boundary conditions

$$U=0,\quad W=0,\quad V=0\quad \text{when}\quad \xi=\pm 1 \tag{2.8}$$

corresponding to (1.1), we obtain two independent systems of homogeneous algebraic equations. From the conditions of solvability of these systems (the principal determinants equal zero) we have the following values for the index λ and the solutions:

– the symmetric problem (U, V, σ_{11}, σ_{22}, σ_{33}, σ_{12} are even functions, and W, σ_{13}, σ_{23} are odd functions with respect to ζ)

$$\lambda_n^a=(2n+1)\frac{\pi}{2},\quad V_a^{(0)}=C_{1n}(\eta)\cos(2n+1)\frac{\pi}{2}\zeta,\quad n\in N \tag{2.9}$$

$$\sin 2\lambda_p = 2\lambda_p, \quad W_p^{(0)} = C_{1p}(\eta)(\sin\lambda_p\zeta - \zeta\,\mathrm{tg}\,\lambda_p\cos\lambda_p\zeta) \tag{2.10}$$

where the subscript a denotes quantities for the antiplane boundary layer and the subscript p denotes quantities for the plane boundary layer; a $\overline{\lambda_p}$ corresponds to each root λ_p of Eq. (2.10), and hence $W_p^{(0)}$ will be real and will contain two unknown real functions of η;

– the skew-symmetric problem (U, V, σ_{11}, σ_{22}, σ_{33}, σ_{12} are odd functions and W, σ_{13}, σ_{23} are even functions of ζ)

$$\lambda_n^a = \pi n, \quad V_a^{(0)} = C_{3n}(\eta)\sin\pi n\zeta, \quad n \in N \tag{2.11}$$

$$\sin 2\lambda_p = -2\lambda_p, \quad W_p^{(0)} = C_{4p}(\eta)(\zeta\sin\lambda_p\zeta - \mathrm{tg}\,\lambda_p\cos\lambda_p\zeta) \tag{2.12}$$

The equations $\sin 2\lambda_p = \pm 2\lambda_p$ have complex-conjugate roots; their values were derived earlier in [5].

When $s > 0$ we need to solve the inhomogeneous equations (2.5) and (2.7), in which case the solution will depend on which of the calculated values λ^a or λ_p is taken as a basis. We need to consider two cases. As a result we will have general solutions of the homogeneous equations of type Q_a and Q_p. If we take λ^a as the basis of the calculations, it will follow from (2.6)–(2.8) that

$$U_a^{(0)} = W_a^{(0)} = \sigma_{11a}^{(0)} = \sigma_{22a}^{(0)} = \sigma_{13a}^{(0)} = \sigma_{33a}^{(0)} \equiv 0 \tag{2.13}$$

If we take λ_p as the basis of the calculations, we will have

$$V_p^{(0)} = \sigma_{12p}^{(0)} = \sigma_{23p}^{(0)} \equiv 0 \tag{2.14}$$

For the approximations $s > 0$ the quantities occurring in formulae (2.13) and (2.14) are non-zero, i.e. the basic antiplane boundary layer will be accompanied by the plane boundary layer, characterized by the same exponential attenuation factor λ^a and vice versa.

Hence, the plane boundary layer and the antiplane boundary layer, for which the quantities have the subscripts a and p respectively, will be called accompanying boundary layers. Note that the quantities for the accompanying boundary layers can be calculated directly if we know the quantities for the basic boundary layers. Solving Eqs (2.5) and (2.7) for $\lambda = \lambda^a$ and $\lambda = \lambda_p$, for the approximations $s > 0$ we will have

$$V^{(s)} = V_{0a}^{(s)} + V_{0p}^{(s)} + V_{*a}^{(s)} + V_{*p}^{(s)}, \quad W^{(s)} = W_{0a}^{(s)} + W_{0p}^{(s)} + W_{*a}^{(s)} + W_{*p}^{(s)} \tag{2.15}$$

where the first two terms in each representation are general solutions of the homogeneous equations (2.5) and (2.7), while the last two are particular solutions of inhomogeneous equations (2.5) and (2.7). According to relations (2.9)–(2.12), on the right-hand sides of Eqs (2.5) and (2.7) there will be trigonometric functions, and hence the determination of the particular solutions will not present any major difficulty. By calculating $V^{(s)}$ and $W^{(s)}$ and, from formula (2.6), $U^{(s)}$, and satisfying conditions (2.8), taking into account the data for the basic boundary layers (2.9)–(2.12), we obtain algebraic systems in the unknowns in the solutions for the accompanying boundary layers (the solutions of the homogeneous equations of the basic boundary layers identically satisfy (2.8)), whence these unknowns are determined uniquely. As a result, we have the following solution for the boundary layers:

– the symmetric problem

$$V^{(s)} = C_{1n}^{(s)}(\eta)\cos(2n+1)\frac{\pi}{2}\zeta + C_{2p}^{(s)}\cos\lambda_p\zeta + V_{*b}^{(s)}, \quad C_{2p}^{(s)} = -\frac{\tilde{V}_{*b}^{(s)}}{\cos\lambda_p}$$

$$W^{(s)} = C_{1p}^{(s)}(\eta)(\sin\lambda_p\zeta - \zeta\,\mathrm{tg}\,\lambda_p\cos\lambda_p\zeta) + C_{1a}^{(s)}(\eta)\sin\lambda_a\zeta + \zeta U_*^{(s)}(\zeta=1)\cos\lambda_a\zeta + W_{*b}^{(s)} \tag{2.16}$$

$$U^{(s)} = \frac{1}{\lambda_p}C_{1p}^{(s)}(\eta)[(\lambda_p - \mathrm{tg}\,\lambda_p)\cos\lambda_p\zeta + \zeta\lambda_p\,\mathrm{tg}\,\lambda_p\sin\lambda_p\zeta] + C_{1a}^{(s)}(\eta)\cos\lambda_a\zeta +$$

$$+\frac{(-1)^n}{\lambda_a}U_*^{(s)}(\zeta=1)(\cos\lambda_a\zeta - \lambda_a\zeta\sin\lambda_a\zeta) + U_*^{(s)}, \quad C_{1a}^{(s)}(\eta) = (-1)^{n+1}\tilde{W}_{*b}^{(s)}$$

$$U_*^{(s)} = \frac{1}{\lambda_a}\frac{\partial W_{*a}^{(s)}}{\partial \zeta} + \frac{1}{\lambda_p}\frac{\partial W_{*p}^{(s)}}{\partial \zeta} + \frac{1}{\lambda_a}\frac{\partial V_a^{(s-1)}}{\partial \eta} + \frac{1}{\lambda_p}\frac{\partial V_p^{(s-1)}}{\partial \eta}$$

$$\lambda_a = (2n+1)\frac{\pi}{2}; \quad \sin 2\lambda_p = 2\lambda_p$$

– the skew-symmetric problem

$$V^{(s)} = C_{3n}^{(s)}(\eta)\sin \pi n\zeta + C_{1p}^{(s)}\sin\lambda_p\zeta + V_{*b}^{(s)}, \quad C_{1p}^{(s)} = -\frac{\tilde{V}_{*b}^{(s)}}{\sin\lambda_p}$$

$$W^{(s)} = C_{4p}^{(s)}(\eta)(\zeta\sin\lambda_p\zeta - \operatorname{tg}\lambda_p\cos\lambda_p\zeta) + (-1)^{n+1}\zeta U_*^{(s)}(\zeta=1)\sin\pi n\zeta +$$
$$+(-1)^{n+1}\tilde{W}_{*b}^{(s)}\cos\pi n\zeta + W_{*b}^{(s)} \tag{2.17}$$
$$U^{(s)} = C_{4p}^{(s)}(\eta)[(1+\lambda_p\operatorname{tg}\lambda_p)\sin\lambda_p\zeta + \zeta\lambda_p\cos\lambda_p\zeta] +$$
$$+\frac{(-1)^{n+1}}{\pi n}U_*^{(s)}(\zeta=1)(\sin\pi n\zeta + \pi n\zeta\cos\pi n\zeta) + (-1)^n\tilde{W}_{*b}^{(s)}\sin\pi n\zeta + U_*^{(s)}$$
$$\sin 2\lambda_p = -2\lambda_p$$

Here

$$V_{*b}^{(s)} = V_{*a}^{(s)} + V_{*p}^{(s)}, \quad \tilde{V}_{*b}^{(s)} = V_{*a}^{(s)}(\zeta=1) + V_{*p}^{(s)}(\zeta=1) \quad (V, W)$$

The remaining quantities are calculated from formulae (2.4) and (2.6).

In formulae (2.16) and (2.17) the functions $C_{1n}^{(s)}(\eta)$ and $C_{3n}^{(s)}(\eta)$ are real while $C_{1p}^{(s)}(\eta)$ and $C_{4p}^{(s)}(\eta)$ are complex. However, since there is a $\overline{\lambda_p}$ corresponding to each λ_p, in the final analysis the expressions for $W^{(s)}$ and $U^{(s)}$ are real and contain two groups of real, for the present arbitrary, functions. In fact, denoting the coefficient of $C_{1p}^{(s)}(\eta)$ in the expression for $W^{(s)}$ by F_w, and that for $U^{(s)}$ by F_u and taking

$$C_{1p}^{(s)}(\eta) = \frac{1}{2}[A_p^{(s)}(\eta) - iB_p^{(s)}(\eta)]$$

we will have

$$C_{1p}^{(s)}(\eta)[\sin\lambda_p\zeta - \zeta\operatorname{tg}\lambda_p\cos\lambda_p\zeta] = A_p^{(s)}\operatorname{Re}F_w + B_p^{(s)}\operatorname{Im}F_w$$
$$C_{1p}^{(s)}(\eta)\frac{1}{\lambda_p}[(\lambda_p - \operatorname{tg}\lambda_p)\cos\lambda_p\zeta + \zeta\lambda_p\operatorname{tg}\lambda_p\sin\lambda_p\zeta] = A_p^{(s)}\operatorname{Re}F_u + B_p^{(s)}\operatorname{Im}F_u \tag{2.18}$$

Hence, the general solution for the antiplane boundary layer and the plane boundary layer contains three groups of, for the present, arbitrary functions, and, as will be shown below, in combination with the solution of the internal problem we can satisfy the three conditions on the side surface of the plate.

In view of the linearity and homogeneity of the equations and boundary conditions, the solution for the boundary layer will also be $\varepsilon^{\chi+s}Q_a^{(s)} + \varepsilon^{\mu+s}Q_p^{(s)}$. The general solution of the problem can then be written in the form

$$Q = Q_{in} + \varepsilon^{\chi+s}Q_a^{(s)} + \varepsilon^{\mu+s}Q_p^{(s)}, \quad s = \overline{0, N} \tag{2.19}$$

where Q_{in} is the solution of the inner problem, and χ and μ are, as yet unknown, integers, which must be chosen so as to obtain a consistent process for satisfying the boundary conditions on the side surface of the plate [5, 9, 10].

When writing solution (2.19), as usual, it was assumed that the quantities for the boundary layer, constructed for $x = 0$, are negligibly small when $x = a$ and vice versa. This imposes limits on the longitudinal dimension of the plate a. Taking the values of the roots (2.9)–(2.12) into account, we have in the symmetric problem

$$1+\exp\left(-\frac{\pi}{2}\frac{a}{h}\right)\approx 1,\quad 1+\exp\left(-\frac{a}{h}\operatorname{Re}\lambda_1\right)\approx 1+\exp\left(-\frac{a}{h}3.75\right)\approx 1 \tag{2.20}$$

and in the skew-symmetric problem

$$1+\exp\left(-\frac{\pi a}{h}\right)\approx 1,\quad 1+\exp\left(-\frac{a}{h}2.11\right)\approx 1 \tag{2.21}$$

In concluding this section, we note that solutions for the boundary layers in the region of $x = a$ and $y = 0, b$ can be constructed in the same way. Data for these boundary layers can be obtained from the data presented above by simple replacement of the variable.

3. MATCHING THE SOLUTION OF THE INNER PROBLEM AND THE SOLUTION FOR THE BOUNDARY LAYER

Using the general solution in the form (2.19), we will describe a procedure for satisfying the boundary conditions on the side surface. Suppose the side surface $x = 0$ is free. It is required to satisfy the following conditions in the three-dimensional problem

$$\sigma_{xx}=\sigma_{xz}=\sigma_{xy}=0 \quad \text{when} \quad x=0 \tag{3.1}$$

By expression (2.19) and the data presented in Sections 1 and 2, conditions (3.1) can be written in the form

$$\begin{aligned}&\varepsilon^{-3+s}\sigma_{xx}^{(s)}+\varepsilon^{\chi-1+s}\sigma_{11a}^{(s)}+\varepsilon^{\mu-1+s}\sigma_{11p}^{(s)}=0\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{when } x=0\ (t=0)\\ &\varepsilon^{-2+s}\sigma_{xy}^{(s)}+\varepsilon^{\chi-1+s}\sigma_{12a}^{(s)}+\varepsilon^{\mu-1+s}\sigma_{12p}^{(s)}=0 \quad (y,z;2,3)\end{aligned}$$

The last conditions will be consistent, i.e. they will enable as to determine in succession quantities both for the inner problem and for the boundary layers if $\chi = \mu = -1$. As a result we have

$$\sigma_{xx}^{(s)}+\sigma_{11a}^{(s-1)}+\sigma_{11p}^{(s-1)}=0;\quad \sigma_{xy}^{(s)}+\sigma_{12a}^{(s)}+\sigma_{12p}^{(s)}=0 \quad (y,z;2,3) \quad \text{when} \quad x=0(t=0) \tag{3.2}$$

Taking relations (2.13) and (2.14) into account, we can write conditions (3.2) when $s = 0$

$$\sigma_{xx}^{(0)}=0;\quad \sigma_{xz}^{(0)}+\sigma_{13p}^{(0)}=0,\quad \sigma_{xy}^{(0)}+\sigma_{12a}^{(0)}=0 \quad \text{when} \quad x=0(t=0) \tag{3.3}$$

Since Eq. (1.9) is of the second order, the first of conditions (3.3) is sufficient to find the solution of the inner problem. After determining $\sigma^{(0)}$ from formulae (1.8) we find $\sigma_{xy}^{(0)}$ and $\sigma_{xz}^{(0)}$. Consequently, to determine the antiplane boundary layer we have the condition

$$\sigma_{12a}^{(0)}(\xi=0)=-\sigma_{xy}^{(0)}(x=0) \tag{3.4}$$

Using relations (2.4) and (2.9) or (2.11), we find the arbitrary constant in the solution for the antiplane boundary layer. Since $\sigma_{xy}^{(0)}\equiv 0$, we have $Q_a^{(0)}\equiv 0$. To determine the plane boundary layer one of the two required conditions follows from conditions (3.3).

$$\sigma_{13p}^{(0)}=-\sigma_{xz}^{(0)}(x=0) \quad \text{when} \quad t=0 \tag{3.5}$$

and the second condition is obtained when considering the approximation $s = 1$.

Taking relations (2.13) into account with $s = 1$ we have the conditions

$$\sigma_{xx}^{(1)}+\sigma_{11p}^{(0)}=0;\quad \sigma_{xy}^{(1)}+\sigma_{12a}^{(1)}+\sigma_{12p}^{(1)}=0 \ (y,z;2,3) \quad \text{when} \quad x=0\,(t=0) \tag{3.6}$$

In Eq. (1.9), $W^{(1)} = 0$, and it follows from formulae (1.8) that

$$\sigma_{xx}^{(1)}=0,\ \sigma_{xz}^{(1)}=0,\quad \sigma_{xy}^{(1)}\neq 0$$

Consequently, the plane boundary layer is found from the conditions

$$\sigma_{11p}^{(0)}=0,\quad \sigma_{13p}^{(0)}=-\sigma_{xz}^{(0)}(x=0)\quad \text{when } t=0 \tag{3.7}$$

From conditions (3.6) we also obtain the condition

$$\sigma_{12a}^{(1)}=-\sigma_{12p}^{(1)}-\sigma_{xy}^{(1)}(x=0)\quad \text{when } t=0 \tag{3.8}$$

for determining the antiplane boundary layer and one of the two conditions for the plane boundary layer

$$\sigma_{13p}^{(1)}=-\sigma_{13a}^{(1)}\quad \text{when } t=0 \tag{3.9}$$

In conditions (3.8) and (3.9) $\sigma_{12p}^{(1)}$ and $\sigma_{13a}^{(1)}$ are known functions, like the quantities of the accompanying boundary layers.

We can similarly combine the solutions for the approximations $s>1$. Note that the procedure described for combining the solutions holds both for the symmetric problem and for the antisymmetric problem.

We will illustrate the procedure described using the following example of an applied nature, which models the operation of a seismic isolator. Suppose the face surfaces of the plate are given constant vertical displacements Δ

$$u_x^{\pm}=u_y^{\pm}=0,\quad u_z^{\pm}=\mp\Delta \tag{3.10}$$

The side surface is load-free, and the change in the temperature field θ is constant.

According to formulae (1.8), $\sigma_{xx}^{(0)}=\sigma_{yy}^{(0)}=\sigma_{zz}^{(0)}=\sigma^{(0)}$, while $\sigma^{(0)}$ is found from Eq. (1.9) with the following right-hand side

$$-\frac{3G}{2h^3}W^{(0)}=\frac{3G}{h^3}(\Delta+3\alpha h\theta)$$

with the first boundary condition of (3.3).

The solution has the form

$$\begin{gathered}\sigma^{(0)}=\sigma=-\sum_{m,n=1}^{\infty}A_{2m-1,2n-1}\sin\frac{\pi(2m-1)x}{a}\sin\frac{\pi(2n-1)y}{b}\\ A_{2m-1,2n-1}=\frac{48G(\Delta+3h\alpha\theta)}{\pi^4h^3}\frac{a^2b^2}{(2m-1)(2n-1)[(2m-1)^2b^2+(2n-1)^2a^2]}\end{gathered} \tag{3.11}$$

According to formulae (1.8), $\sigma_{xy}^{(0)}=0$, and it follows from relations (2.4) and (2.9) that $V_a^{(0)}=0$, i.e. $Q_a^{(0)}=0$. Hence, the antiplane boundary layer can be neglected.

To determine the plane boundary layer we have the boundary conditions

$$\begin{gathered}\sigma_{11p}^{(0)}=0\quad \text{when } t=0\\ \sigma_{13p}^{(0)}=-\sigma_{xz}^{(0)}(x=0)=-\frac{\pi l}{a}\zeta\sum_{m,n=1}^{\infty}A_{2m-1,2n-1}(2m-1)\sin\frac{\pi(2n-1)}{b}y\end{gathered} \tag{3.12}$$

Using relations (2.6), (2.10) and (2.18), satisfying these conditions, we determine the plane boundary layer. Note that conditions (3.12) can only be satisfied approximately, for which we can use the boundary collocation method, the method of least squares, etc.

When $s=1$

$$\sigma_{xx}^{(1)}=\sigma_{yy}^{(1)}=\sigma_{zz}^{(1)}=\sigma_{xz}^{(1)}=\sigma_{yz}^{(1)}=0,\quad \sigma_{xy}^{(1)}=h^2(1-\zeta^2)\frac{\partial^2\sigma}{\partial x\partial y}$$
$$u_x^{(1)}=u_y^{(1)}=u_z^{(1)}=0 \tag{3.13}$$

At this stage the antiplane boundary layer is determined from condition (3.8) and relations (2.4) and (2.16) by obvious operations.

When $s = 2$

$$\sigma_{xx}^{(2)}=h^2(1-\zeta^2)\frac{\partial^2\sigma}{\partial x^2}-\frac{3}{2h}G\zeta^2(\Delta+3\alpha h\theta),\quad \sigma_{xz}^{(2)}=\sigma_{yz}^{(2)}=\sigma_{xy}^{(2)}=0$$
$$u_x^{(2)}=u_y^{(2)}=u_z^{(2)}=0 \tag{3.14}$$

To determine the boundary layers we have from (3.2) the conditions

$$\sigma_{11p}^{(1)}=-\sigma_{11a}^{(1)}-\sigma_{xx}^{(2)}(x=0),\quad \sigma_{13p}^{(1)}=-\sigma_{13a}^{(1)},\quad \sigma_{12a}^{(2)}=-\sigma_{12p}^{(2)}\quad \text{when } t=0 \tag{3.15}$$

On the right-hand sides of (3.15) there are known functions, but in practical applications it is hardly necessary in calculations of higher approximations for the boundary layer.

When $s \geqslant 3$ we have $Q_{in}^{(s)} \equiv 0$, and the process of determining the quantities of the internal problem is terminated. Combining all the approximations, we will have for the quantities of the inner problem

$$\sigma_{xx}=\sigma+(h^2-z^2)\frac{\partial^2\sigma}{\partial x^2}-\frac{3G}{2h^3}z^2(\Delta+3\alpha h\theta)\,(x,y),\quad \sigma_{zz}=\sigma+\frac{3G}{2h^3}z^2(\Delta+3\alpha h\theta)$$
$$\sigma_{xy}=(h^2-z^2)\frac{\partial^2\sigma}{\partial x\partial y},\quad \sigma_{xz}=-z\frac{\partial\sigma}{\partial x}\,(x,y) \tag{3.16}$$
$$u_x=\frac{1}{2G}(h^2-z^2)\frac{\partial\sigma}{\partial x}\,(x,y),\quad u_z=3z\alpha\theta+\frac{z}{2h^3}(z^2-3h^2)(\Delta+3\alpha h\theta)$$

From the formulae obtained for the displacements we show in Fig. 1 the deformed state of the incompressible plate. In view of the symmetry we only show a half of the plane.

This example models the operation of rubber–metallic seismic isolators [11–15]. It was also considered earlier in [15] based on certain hypotheses and ignoring the temperature field; the solution obtained in [15] naturally differs somewhat from the asymptotically exact solution (3.16).

4. THE SOLUTION OF THE INNER PROBLEM FOR A THREE-LAYER PLATE WITH A MIDDLE INCOMPRESSIBLE LAYER

The general solution constructed above for an incompressible plate and the solution for a compressible plate [22, 5] together enable us to obtain the stress–strain state of multilayered plates, some of the layers of which are made of compressible material and the others of incompressible material. We will present an algorithm of the solution and data for three-layer plates with a middle incompressible layer, but the approach remains valid for multilayered plates also.

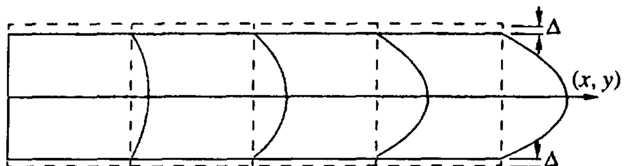

Fig. 1

Consider a three-layer plate, which occupies the volume

$$\Omega = \{x, y, z: 0 \leqslant x \leqslant a, 0 \leqslant y \leqslant b, -h - h_e \leqslant z \leqslant h + h_e, \max\{h, h_e\} < \min\{a, b\}\}$$

where the layers $h_e \leqslant z \leqslant h + h_e$, $-h - h_e \leqslant z \leqslant -h_e$ are made of compressible material while the middle layer $-h_e \leqslant z \leqslant h_e$ is made of incompressible material.

The following kinematic conditions are given on the face surfaces of the plate

$$u_j[z = \pm(h + h_e)] = u_j^{\pm}(x, y), \quad j = x, y \tag{4.1}$$

The following complete contact conditions are satisfied between the layers

$$\begin{aligned} &u_j^{(1)}(z = h_e) = u_j^e(z = h_e), \quad \sigma_{jz}^{(1)}(z = h_e) = \sigma_{jz}^e(z = h_e) \\ &u_j^{(3)}(z = -h_e) = u_j^e(z = -h_e), \quad \sigma_{jz}^{(3)}(z = -h_e) = \sigma_{jz}^e(z = -h_e), \quad j = x, y, z \end{aligned} \tag{4.2}$$

The temperature field is specified by the function $\theta(x, y, z)$.

Here and henceforth quantities relating to the first and third layers are given the superscripts (1) and (3), while quantities relating to the middle layer are given the superscript e.

The components of the stress tensor and the displacement tensor are calculated for the incompressible layer from formulae (1.4)–(1.7), while for the compressible layers they are calculated from formulae derived previously in [2], and conditions (4.1) and (4.2) are satisfied directly. As a result we obtain an iteration process for determining all the required quantities with an asymptotic accuracy specified in advance. The method described above was also used to construct the boundary layer, but the calculations and the final formulae are extremely lengthy, and hence we will only give the data for the inner problem here.

If we confine ourselves to the first two steps of the iteration and introduce the notation

$$t_j^- = \frac{u_j^+ - u_j^-}{2}, \quad t_j^+ = \frac{u_j^+ + u_j^-}{2}, \quad j = x, y, z, \quad P = \frac{h_e}{G_e}, \quad M = \frac{h}{G}, \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

then, for a three-layer plate with a constant change in the temperature field we will have:

– for the first (compressible) layer $h_e \leqslant z \leqslant h_e + h$

$$\begin{aligned} &\sigma_{zz}^{(1)} = \sigma, \quad \sigma_{xx}^{(1)} = \sigma_{yy}^{(1)} = \frac{\nu}{1-\nu}\sigma - \frac{2(1+\nu)}{1-\nu}G\alpha\theta, \quad \sigma_{xy}^{(1)} = 0 \\ &\sigma_{xz}^{(1)} = \frac{1}{P+M}\left[t_x^- + h\frac{\partial t_z^+}{\partial x}\right] - \left(h_e\frac{1-2\nu}{1-\nu} + z\frac{\nu}{1-\nu}\right)\frac{\partial \sigma}{\partial x} \quad (x, y) \\ &u_x^{(1)} = u_x^+ - (z - h - h_e)\frac{\partial u_z^+}{\partial x} + \frac{M}{h}(z - h - h_e)\left[\frac{1}{P+M}\left(t_x^- + h\frac{\partial t_z^+}{\partial x}\right) - h_e\frac{1-2\nu}{1-\nu}\frac{\partial \sigma}{\partial x}\right] + \\ &+\frac{M}{h}\left(\frac{(h+h_e)^2 - z^2}{2}\frac{\nu}{1-\nu} - \frac{(z-h-h_e)^2}{4}\frac{1-2\nu}{1-\nu}\right)\frac{\partial \sigma}{\partial x} \quad (x, y) \\ &u_z^{(1)} = u_z^+ + (z - h - h_e)\frac{M}{2h}\frac{1-2\nu}{1-\nu}\sigma + (z - h - h_e)\frac{1+\nu}{1-\nu}\alpha\theta \end{aligned} \tag{4.3}$$

– for the middle (incompressible) layer $-h_e \leqslant z \leqslant h_e$

$$\sigma_{zz}^e = \sigma, \quad \sigma_{xx}^e = \sigma_{yy}^e = \sigma, \quad \sigma_{xy}^e = 0$$

$$\sigma_{xz}^e = \frac{1}{P+M}\left[t_x^- + h\frac{\partial t_z^+}{\partial x}\right] - z\frac{\partial \sigma}{\partial x} \quad (x, y)$$

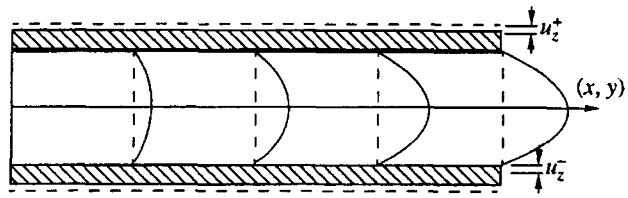

Fig. 2

$$u_x^e = t_x^+ + h\frac{\partial t_z^-}{\partial x} + \frac{P}{P+M}\frac{z}{h_e}\left(t_x^- + h\frac{\partial t_z^+}{\partial x}\right) +$$

$$+\left[\frac{Ph_e}{2} - \frac{P}{2h_e}z^2 + M\left(h_e + \frac{h}{4}\frac{4\nu-1}{1-\nu}\right)\right]\frac{\partial\sigma}{\partial x} \quad (x,y) \tag{4.4}$$

$$u_z^e = t_z^+ + 3z\alpha_e\theta - \frac{P}{P+M}\frac{(h_e^2 - z^2)}{2h_e}\left(\frac{\partial t_x^-}{\partial x} + \frac{\partial t_y^-}{\partial y} + h\nabla^2 t_z^+\right) - z\left(\frac{\partial t_x^+}{\partial x} + \frac{\partial t_y^+}{\partial y} + h\nabla^2 t_z^-\right) -$$

$$-\left[\frac{Ph_e}{2}z - \frac{P}{6h_e}z^3 + M\left(h_e + \frac{h}{4}\frac{4\nu-1}{1-\nu}\right)z\right]\nabla^2\sigma$$

– for the third (compressible) layer $-h-h_e \leqslant z \leqslant -h_e$ the solution is given by formulae which differ from (4.3) only by replacing h_e by $-h_e$ and $h + h_e$ by $-(h + h_e)$.

Here σ is the solution of the equation

$$\nabla^2\sigma - k^2\sigma = -k^2\Phi \tag{4.5}$$

$$k^2 = \frac{1-2\nu}{1-\nu}\left[2h_e\left(h_e + \frac{h}{4}\frac{4\nu-1}{1-\nu}\right) + \frac{2P}{3M}h_e^2\right]^{-1}$$

$$\Phi = \frac{2(1-\nu)}{M(1-2\nu)}\left[t_z^- + hh_e\nabla^2 t_z^- + h_e\left(\frac{\partial t_x^+}{\partial x} + \frac{\partial t_y^+}{\partial y}\right) - \left(3h_e\alpha_e + h\frac{1+\nu}{1-\nu}\alpha\right)\theta\right]$$

when $x = 0$ which integrally satisfies the free-edge conditions. This solution, when $\theta = \text{const}$, has the form

$$\sigma = \sum_{m,n=1}^{\infty} A_{mn}\sin\frac{\pi mx}{a}\sin\frac{\pi ny}{b} + B$$

$$A_{mn} = \frac{4k^2}{ab}\left[\left(\frac{\pi m}{a}\right)^2 + \left(\frac{\pi n}{b}\right)^2 + k^2\right]^{-1}\int_0^a\int_0^b (\Phi - B)\sin\frac{\pi mx}{a}\sin\frac{\pi ny}{b}\,dxdy \tag{4.6}$$

$$B = \frac{2h(1+\nu)G\alpha\theta}{h\nu + h_e(1-\nu)}$$

The deformed state of the three-layer plate is represented in Fig. 2.

The problem considered is, in particular, a model problem for designing rubber–metal seismic isolators [15], and the accuracy assumed to obtain solution (4.3), (4.4), (4.6) is sufficient for practical calculations.

In conclusion we note that the asymptotic method employed also enables one to consider different classes of problems for multilayered plates, including dynamic problems.

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